Resurgence and asymptotic analysis

Path integrals are a very useful way to think about quantum mechanics (QM) due to Dirac and Feynman. Some of its famous benefits are that path integrals generalizes nicely to quantum field theory, and make the connection between quantum and classical physics clearer. In these exercises, we'll explore another benefit of path integrals: they make it conceptually easier to understand the non-perturbative behavior of interacting QM systems, and give insight into some universal features of perturbation theory.

But, despite their benefits, QM path integrals are mathematically-fearsome objects. For example, the path integral used for describing the time-evolution of a one-dimensional quantum system is an infinite-dimensional integral with an oscillatory integrand, looking roughly like

$$\int d\{q(t)\}\, e^{\frac{i}{\hbar}S[q(t)]}$$

where $q(t)$ is the position of a particle and $S[q(t)]$ is the action for the system. For instance, for an anharmonic oscillator, $S[q(t)] = \int dt \frac{1}{2}\dot{q}(t)^2 - 1/2\omega^2 q(t)^2 - \lambda q(t)^4$ (with the mass set to 1).

So, rather than attacking path integrals directly, here let's examine the behavior of a much simpler related integral

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq\, e^{-S} \tag{1}$$

where

$$S = \frac{1}{2}q^2 + \lambda q^4 \tag{2}$$

To study anharmonic oscillators in QM, we do perturbation theory in the parameter λ, assuming that it is small. Let's do that for $Z(\lambda)$, and study the structure of the perturbative expansion. Some formulas which might be useful in the calculations are given after the questions.

1. What is the value of Z when $\lambda = 0$?

2. Perturbation theory represents some complicated quantity, in our case $Z(\lambda)$, by a power series in λ

$$Z(\lambda) = \sum_{n=0}^{\infty} p_n \lambda^n \tag{3}$$

   What is the coefficient p_n of λ^n in the perturbative expansion of $Z(\lambda)$? Hint: try to Taylor-expand the integrand.

3. What is the large-n limit of $|p_n \lambda^n|$? Show that at large n, one can approximate p_n by

$$p_n \approx \frac{(n-1)!}{\pi} \frac{1}{S_*^n} 2^{-1/2}, \qquad S_* = -1/16 \tag{4}$$

4. For a series to converge, $|p_n \lambda^n|$ should approach zero as $n \to \infty$. What does $|p_n \lambda^n|$ do here? Compute the minimum, n_*, of $|p_n \lambda^n|$ as a function of n. At the minimum, you should find that $|p_n \lambda^n| \approx \frac{4e^{-\frac{1}{16\lambda}}\sqrt{\lambda}}{\sqrt{\pi}}$.

5. Confirm your analytic results by plotting $|p_n \lambda^n|$ as a function of n for e.g. $\lambda = 1/100$. You should be finding that for a fixed small λ, as n increases the terms in Eq. (3) initially decrease, reach a minimum at e.g. $\lceil n_* \rceil = 7$ with $\lambda = 1/100$, and then blow up, making the series divergent.

6. Calculate the saddle points of the action S, and show that the value of the action at these saddle points is $S_0 = 0$ and $S_* = -\frac{1}{16}$.

Perturbation theory, both in our toy integral and in QM path integrals, is actually an expansion in fluctuations around the saddle points. In fact, when setting up perturbation theory for Eq. (3), you essentially summed up the fluctuations around the 'perturbative' saddle $S_0 = 0$. The result was a divergent series, which is also typical in QM perturbation theory. It also happens in quantum field theory and in string theory.

This may look extremely alarming, but the divergences of perturbation theory are not at all random! They encode vitally important non-perturbative information about the physics and mathematics of a system. The idea of resurgence theory is to take this seriously and decode this information. For instance, in the example in this exercise, perturbation theory around the saddle $S_0 = 0$ somehow knew that that there was another saddle with the action $S_* = -1/16\lambda$ as one can see from Eq. (4).

Useful formulas:

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2/2} = \sqrt{2\pi} \tag{5}$$

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2/2} x^n = \frac{\sqrt{\pi}\, 2^{-\frac{n}{2}-\frac{1}{2}} \left[(-1)^n + 1\right] n!}{\left(\frac{n}{2}\right)!} \tag{6}$$

$$n! \approx \sqrt{2\pi n}\, e^{n(-1+\log n)} \tag{7}$$